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PRESSURE VESSELS WITH EDGE ZONE YIELDING*

C. R. STEELE

Department of Aeronautics and Astronautics, Stanford University

Abstract—Solutions are obtained for the one-time pressurization and heating of a thin shell of revolution composed of a nonlinear elastic material and clamped to a rigid ring. The usual "edge effect" approximate equations are used. A bilinear moment–curvature relation is considered which indicates that, before the collapse pressure indicated by limit analysis is reached, significant yielding occurs in a narrow edge zone, within the usual "edge zone", for materials with low strain hardening. With the assumption that such an edge zone generally occurs, a simple approximate solution is obtained for any prescribed moment–curvature relation, such as that obtained for a real material after the interaction with the direct stress is taken into consideration. The analysis should be of utility for vessels made of materials of moderate ductility for which failure occurs at the edge region at a pressure less than the collapse pressure.

INTRODUCTION

THE typical pressure vessel is a thin-walled shell of revolution. It is quite well-known that the stress in such a vessel is essentially given by the "membrane" solution except in the vicinities of stiffening rings, or any other geometric discontinuity, where additional stresses occur due to the "edge effect" solutions. See, for instance, the discussion of the linear theory by Gol'denveizer [1] and Flügge [2]. However, for a minimum weight vessel made of a ductile material to be loaded only one time, the plastic limit analysis procedure, discussed by Hodge [3] and applied to a more complex problem by Ruiz and Chukwujekwu [4], is quite successful. Limit analysis indicates that for short cylinders, for instance, end rings provide a significant stiffening effect. However, for moderately long cylinders collapse occurs when the membrane stress is in negligible excess of the yield stress [3]. Instead of a rigid–perfectly plastic material behavior, Constantino *et al.* [5] use a rigid–linear strain hardening material model. They also find the result that plastic collapse occurs when the membrane stress is equal to or in excess of, for short cylinders, the yield stress. However, the linear elastic solution indicates that the discontinuity stresses, in some cases, may far exceed the membrane stress. Thus, except for very ductile materials, failure can occur at the discontinuity regions while the main portion of the shell is well below the yield stress.

The objective of the present investigation is a simple failure estimate for pressure vessels made of moderately ductile materials. First, the essential features of the well-known linear elastic analysis are given for a pressure vessel clamped to a rigid ring. Since the linear solution indicates that the edge bending stress is dominant, a bilinear relation between meridional bending moment and curvature change is assumed. A solution is found which gives a small region of yielding near the edge characterized by high curvature but essentially constant displacement. The assumption that a similar yield zone would generally occur enables a simple solution to be obtained for the actual post yielding moment–curvature relation for a given material.

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In this paper only the material nonlinearity is considered. However, it appears that for a class of shell problems, including the present one, that all the complicating effects of material and geometric nonlinearities, as well as shear deformation, are significant in a narrow zone within the edge zone of the linear bending effect. This effect of the geometric nonlinearity was discussed by Reissner [6]. Thus a direct analysis of a shell theory including all these effects, for instance that given by Green *et al.* [7], would appear to be fruitful. The narrowness of the zone, and hence nearly constant displacement despite a large curvature change, would provide the simplifying feature for a suitable perturbation analysis. Although such a formal perturbation expansion would be logically satisfying, the details tend to obscure the basic ideas involved and, therefore, will be omitted from this paper.

LINEAR SOLUTION

The pressurized and heated shell of revolution indicated in Fig. 1 has the membrane solution

$$N_\varphi^m = pr_2/2 \tag{1a}$$

$$N_\theta^m = N_\varphi^m(2-r_2/r_1) \tag{1b}$$

where the familiar notation of [2] is used. The radial deflection of the membrane solution is

$$h^m = \frac{pr_2 r}{2Et}\left[2-\frac{r_2}{r_1}-\nu\right]+r\alpha T \tag{1c}$$

and the rotation is $\chi^m = O(h^m/r)$. The $r\alpha T$ term gives the expansion due to the known temperature distribution. For simplicity the case of a rigid ring clamped to the shell edge is considered, for which the boundary conditions for the "edge effect" solution are

$$w = -h^m/\sin\varphi$$

$$\chi = -\chi^m$$

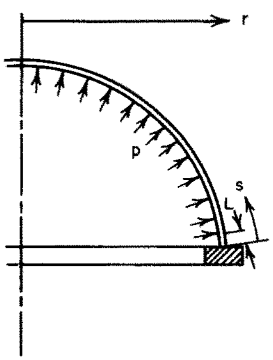

FIG. 1. Pressure vessel attached to ring.

The equation for the edge effect solution, which are obtained from the rigorous equations ([2], p. 359) with the assumption that only the highest derivative terms are significant, may be reduced to

$$\begin{aligned} m''+4\eta &= 0 \\ m &= \eta'' \end{aligned} \qquad (2)$$

where primes denote differentiation with respect to

$$\zeta = [3(1-\nu^2)]^{\frac{1}{4}}\frac{s}{(r_2 t)^{\frac{1}{2}}}$$

in which s is the meridional arclength. The equation (2) is equivalent to the equation (8.6) on page 488 of [1]. The quantities m and η are the dimensionless moment and displacement

$$\begin{aligned} m &= M_\varphi/M_y \\ \eta &= w\left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 Et^3/[12(1-\nu^2)M_y] \end{aligned}$$

where M_y, for this section is some arbitrarily selected value of the moment, but in anticipation of the next section will be identified as the yield moment of the material. The solution of (2) is

$$\eta = \mathrm{e}^{-\zeta}[A_5\cos\zeta + A_6\sin\zeta]$$

where the constants are determined from the edge conditions

$$\begin{aligned} A_5 &= \eta(0) = -\rho/2 \\ &= -\frac{h^m}{\sin\varphi}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 Et^3/[12(1-\nu^2)M_y] \\ A_6 &= A_5\{1+O[(t/r)^{\frac{1}{2}}]\} \\ &\approx A_5 \end{aligned}$$

in which the parameter ρ is introduced, which gives the edge bending stress

$$\sigma_{\varphi B} = \frac{6M_\varphi}{t^2} \approx \left(\frac{3}{1-\nu^2}\right)^{\frac{1}{2}} h^m/r = \left(\frac{3}{1-\nu^2}\right)^{\frac{1}{2}}\frac{N_\varphi^m}{t}\left(2-\frac{r_2}{r_1}-\nu\right).$$

The ratio of maximum bending stress to meridional membrane stress is (for $\nu = 0.3$)

$$\frac{\sigma_{\varphi B}}{\sigma_{\varphi D}} = \begin{cases} 1.27 \text{ for sphere } (r_2/r_1 = 1) \\ 3.1 \quad \text{for cone or cylinder } (r_1 = \infty) \end{cases}$$

which gives the ratio of maximum edge stress to maximum membrane stress

$$\frac{\sigma^e}{\sigma^m} = \begin{cases} 2.27 \text{ for sphere} \\ 2.05 \text{ for cone or cylinder.} \end{cases}$$

Thus, for a shell wall thick enough for the edge stress to be just equal to the yield stress, the interior membrane stress will be substantially less than yield. Since most of the shell is at the membrane stress, such a design may seem unreasonably conservative. The situation is alleviated by a variable wall thickness; but in some situations this could present worse manufacturing problems. Flexibility of the ring somewhat reduces the edge stress but does not change the features of the problem. Therefore a solution for the shell clamped to a rigid ring with pressure in excess of that which produces incipient yielding is obtained in the following section.

SOLUTION FOR EDGE YIELD ZONE

The linear solution indicates that, as the pressure is increased, the meridional stress at the edge first reaches the yield stress. Furthermore, since $N_\varphi \approx N_\varphi^m$, $\nu N_\varphi^m \leqq N_\theta \leqq N_\theta^m$, $M_\theta \approx \nu M_\varphi$, it is the relation between M_φ and the curvature change that is most significantly affected by material yielding. Thus the usual linear constitutive relations are assumed except for the moment–curvature relation.

Bilinear moment–curvature relation

First is considered the bilinear relation

$$m = \begin{cases} \eta'' & \text{for } |\eta''| < 1 \\ 1+\alpha^{-4}(\eta''-1) & \text{for } \eta'' > 1 \end{cases} \tag{3}$$

which is shown in Fig. 2. The quantity M_y is now identified as the yield moment. Then ρ is the ratio of pressure to the pressure causing incipient yield

$$\rho = p/p_y.$$

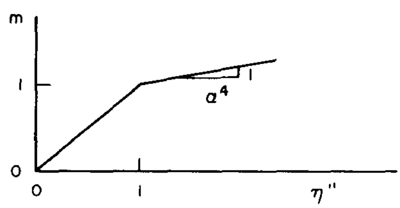

FIG. 2. Moment–curvature relation.

Since, as seen from, for instance, the discussion of [1], only the derivative of M_φ is significant for the edge effect equations which now reduce to

$$\eta'''' \times \begin{Bmatrix} \alpha^{-4} \text{ for } \eta'' > 1 \\ 1 \quad \text{for } |\eta''| < 1 \end{Bmatrix} + 4\eta = 0. \tag{4}$$

Solutions for $\eta'' > 1$ are of the form

$$\eta = \exp[(-1)^{\frac{1}{4}} 2^{\frac{1}{2}} \alpha\zeta].$$

The solution will be sought for which $\eta'' > 1$ in a zone near the edge $0 \leqq s \leqq L$, and $|\eta''| < 1$ for $s > L$. Such a solution would be

$$\eta = \begin{cases} \cosh\alpha(\zeta-l)[A_1\cos\alpha(\zeta-l)+A_2\sin\alpha(\zeta-l)] & \\ +\sinh\alpha(\zeta-l)[A_3\cos\alpha(\zeta-l)+A_4\sin\alpha(\zeta-l)] & \text{for } 0 \leqq \zeta \leqq l \\ e^{-(\zeta-l)}[A_5\cos(\zeta-l)+A_6\sin(\zeta-l)] & \text{for } \zeta \geqq l \end{cases} \tag{5}$$

where $l = \zeta(L)$ and where the A_i are constants to be determined from the conditions at the edge on displacement and rotation

$$-\eta(0) = \rho/2$$
$$\eta'(0) = O(\rho/r\zeta') \approx 0$$

and the continuity and yield conditions at $\zeta = l$. Since displacement and slope are continuous, η and $\mathrm{d}\eta/\mathrm{d}\zeta$ are continuous, but for continuity of the transverse shear

$$\eta'''(l^+) = \alpha^{-4}\eta'''(l^-).$$

The yield condition is

$$\eta''(l^+) = \eta''(l^-) = 1$$

The condition at $\zeta = l$ give

$$A_1 = A_5$$
$$\alpha^2 A_4 = -A_6 = \tfrac{1}{2}$$
$$\alpha(A_2 + A_3) = A_6 - A_5$$
$$\alpha^{-1}(A_2 - A_3) = A_5 + A_6.$$

Then the conditions at $\zeta = 0$ give

$$-2A_5[\mathrm{Cc} - \tfrac{1}{2}(\alpha - \alpha^{-1})\mathrm{Cs} + \tfrac{1}{2}(\alpha + \alpha^{-1})\mathrm{Sc}]$$
$$= \rho + \tfrac{1}{2}(\alpha + \alpha^{-1})\mathrm{Cs} - \tfrac{1}{2}(\alpha - \alpha^{-1})\mathrm{Sc} + \alpha^{-2}\mathrm{Ss} \tag{6a}$$
$$2A_5[-\alpha^{-1}\mathrm{Cc} + \mathrm{Cs} - \mathrm{Sc} + \alpha\mathrm{Ss}] = \alpha^{-1}\mathrm{Cc} + \alpha^{-2}\mathrm{Cs} + \alpha^{-2}\mathrm{Sc} + \alpha\mathrm{Ss} \tag{6b}$$

in which

$$\mathrm{C} = \cosh \alpha l \qquad \mathrm{S} = \sinh \alpha l$$
$$\mathrm{c} = \cos \alpha l \qquad \mathrm{s} = \sin \alpha l.$$

The two equations (6a, b) must be solved for the two unknowns A_5 and l. Generally, the solution would have to be obtained numerically. However, approximations can be obtained which give the interesting features of the solution.

For $\alpha l \ll 1$ one obtains from (6b)

$$-2A_5 = \frac{(\alpha^{-1} + 2\alpha^{-1}l + \alpha^3 l^2)[1 + O((\alpha l)^4)]}{(\alpha^{-1} - \alpha^3 l^2)[1 + O(\alpha^4 l^3)]} \tag{7a}$$
$$= 1 + 2l + 2\alpha^4 l^2 + \ldots . \tag{7b}$$

From the first equation (6a) the result is obtained

$$\rho = 1 + 2l + (2\alpha^4 + 1)l^2 + \ldots$$

which may be inverted to give the width of the edge zone as a function of the load

$$l = \frac{\rho - 1}{2} - \frac{\alpha^2}{8}(2\alpha^2 + \alpha^{-2})(\rho - 1)^2 + O((\rho - 1)^3) \tag{8a}$$

The curvature at the edge is

$$\eta''(0) = \alpha^2[A_4\mathrm{Cc}+A_3\mathrm{Cs}-A_2\mathrm{Sc}-A_1\mathrm{Ss}]$$
$$= 1+\alpha^4(\rho-1)-\tfrac{1}{2}\alpha^4(\alpha^4-1)(\rho-1)^2+O((\rho-1)^3). \tag{8b}$$

The three terms provide a valid solution for either pressure only slightly in excess of the yield pressure $\rho-1 \ll 1$ and/or for a small change in the slope of the moment–curvature relation $|\alpha^4-1| \ll 1$. In particular when $\alpha = 1$, the linear result is obtained $\eta''(0) = \rho$. The three terms indicate that the curve for ρ as a function of $\eta''(0)$ begins with the same slope as in Fig. 2 at $\rho = 1$ but then the slope increases as shown in Fig. 3. The curve for $\alpha^4 = 2$ is computed from (8b).

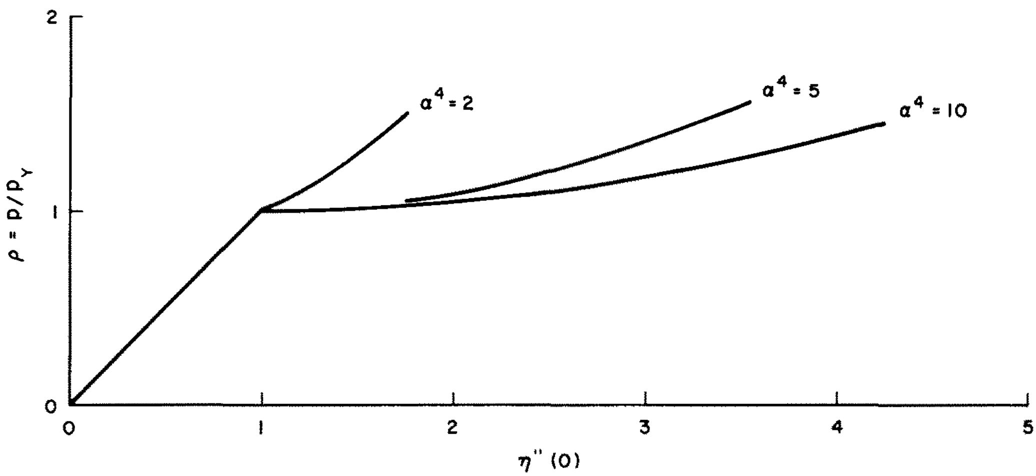

FIG. 3. Relation between load and edge curvature.

An approximation for the solution for large values of $\rho-1$ can be obtained for $\alpha \gg 1$. The second term of the expansion (8a) at least indicates that l becomes small as α or ρ become large. Indeed the correct solution is $l = O(\alpha^{-2})$ for which (7a) is valid but not (7b), which is replaced by

$$-2A_5 \approx \frac{1+(\alpha^2 l)^2}{1-(\alpha^2 l)^2}.$$

Then (6a) gives $A_5 \approx -\rho$ so that

$$\alpha^2 l \approx \left(\frac{\rho-1}{\rho+1}\right)^{\frac{1}{2}} \tag{9a}$$

and

$$\eta''(0) \approx 1+\alpha^4 l(1+A_5)$$
$$\approx 1+\alpha^2(\rho^2-1)^{\frac{1}{2}}. \tag{9b}$$

This expression gives the curves of Fig. 3 for $\alpha^4 = 5$ and 10.

Thus, for a material with little strain hardening past yield, giving a large value of α^4, a pressure load slightly in excess of the incipient yield load causes a relatively large curvature at the edge. Furthermore (9a) indicates that the width of the yielding zone remains small as the load is increased. Also note that $\eta(l) \equiv A_5 \approx -\rho/2 \equiv \eta(0)$. Thus the yield zone is a narrow region of relatively large curvature but of almost constant displacement. A yield zone of similar properties would be expected for, instead of the bilinear relation Fig. 2, a moment–curvature curve something like that of Fig. 4 for an actual material.

General moment–curvature relation

A simple solution for the actual material can be obtained from the assumption that the yield zone is sufficiently narrow $l \ll 1$ so that the displacement in the zone is essentially constant. The equations are

$$m''+4\eta = 0 \qquad m = \begin{cases} \eta'' & \text{for } l \leqq \zeta \\ f(\eta'') & \text{for } 0 \leqq \zeta \leqq l. \end{cases} \tag{10}$$

The conditions are

$$\eta(0) = -\rho/2 \approx \eta(l)$$
$$\eta''(l) = 1$$
$$\eta'(0) = 0.$$

In the zone $l \leqq \zeta$, the solution is

$$\eta = e^{-(\zeta-l)}[A_5\cos(\zeta-l)+A_6\sin(\zeta-l)]$$

where the constants are

$$A_5 \approx -\rho/2$$
$$A_6 = -\tfrac{1}{2}$$

from which is obtained

$$\eta'(l) = (\rho-1)/2$$
$$\eta'''(l) = -(\rho+1)$$

For the yield zone $0 \leqq \zeta \leqq l$

$$\eta \approx -\rho/2$$

so that two integrals of the equilibrium equation (10) give

$$f(\eta'') = m \approx \rho(l-\zeta)^2+(\rho+1)(l-\zeta)+1 \tag{11}$$

where the coefficient of the linear term is chosen to make the transverse shear m' continuous at $\zeta = l$. A typical curve for m as a function of $l-\zeta$ is shown by the dashed line in Fig. 4. So (11) can be easily inverted, at least numerically or graphically, to provide the relation, for $0 \leqq \zeta \leqq l$,

$$\eta'' = g(l-\zeta)$$

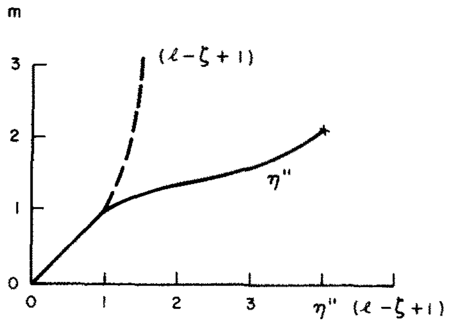

FIG. 4. Typical moment–curvature relation.

a typical curve of which is shown in Fig. 5. An integration gives

$$\eta'(0) = \int_l^0 g(l-\zeta)\,\mathrm{d}\zeta + \eta'(l)$$

that is

$$(\rho-1)/2 = \int_0^l g(x)\,\mathrm{d}x.$$

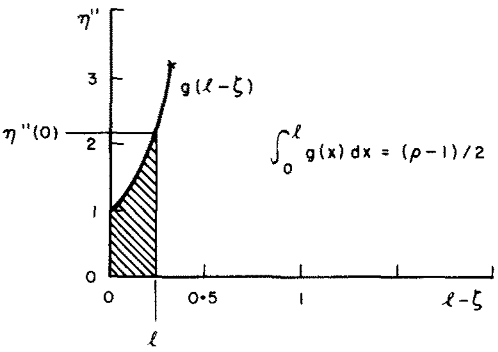

FIG. 5. Yield zone curvature.

Thus a prescribed value of ρ, as well as the curve $f(\eta'')$, determines the curve $g(x)$. Then l, the length of the yield zone, and the edge curvature are obtained from a single integration. If l turns out to be small in comparison with unity, then the result should be valid. For the real material, the constitutive relations become quite complex for the nonlinear range. However, for the pressure vessel clamped to the rigid ring, the circumferential strain is zero at the edge and so, should be small in the narrow yield zone. Hence, in this zone the shell moment–curvature relation Fig. 4 is obtained just as for a beam and will depend on the local temperature and the interaction with the longitudinal membrane strain N_φ^m/ET which is less than the yield strain and is essentially constant in the edge zone. The inner surface of the shell wall will reach the failure strain at some value of curvature indicated by the point x in Fig. 4. If the maximum curvature $\eta''(0)$ from Fig. 5 is less than the failure curvature, then the material has sufficient ductility to take the edge bending.

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Абстракт—Определяются решения для тонкой оболочки вращения при одновременном давлении и нагреве, изготовленной из нелинейного, упругого материала, и защемленной в жесткои кольце. Для обыкновенного "краевого эффекта" используются приближенные уравнения. Исследуется зависимость: билинейный момент-кривизна, которая указывает что прежде чем появится разрушение давлением, указанное анализом предельного состояния, возникает имеющие значение течение в узкой краевой зоне, без обыкновенной "краевой зоны", для материалов со слабым упрочнением. Учитывая, что такая краевая зона вообще появляется, получается простое решение для произвольной предписанной зависимости: момент-кривизна, такое как для действительного материала, когда принимается во внимание реакции с простым напряжением. Анализ можно использовать для резервуаров, изготовленных из материалов, обладающих средней пластичностью, для которых разрушение появляется в районе края при давлении реже, чем разрушение давлением целой конструкции.